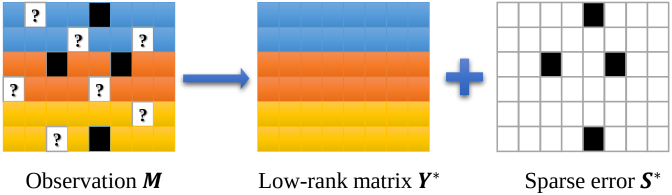
Matrix Completion

- Given partial and corrupted observation M, decomposing it into the sum of two matrices: (1) Low-rank matrix $\boldsymbol{X}^*$, and (2) Sparse error $\boldsymbol{S}^*$;



- Optimization problem:

$$\min_{\boldsymbol{X},\boldsymbol{S}} : \quad \|\mathcal{P}_\Omega(\boldsymbol{M} - \boldsymbol{X} - \boldsymbol{S})\|_F^2$$

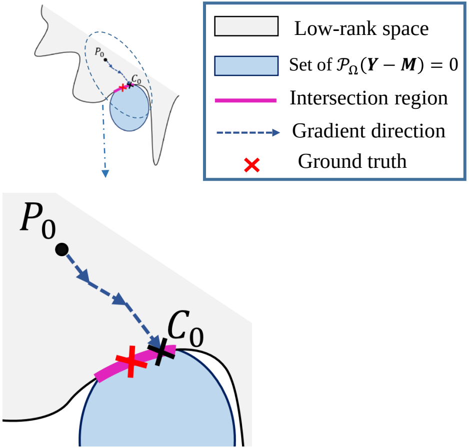
$$\text{s.t.} \quad \text{rank}(\boldsymbol{X}) \leq r, \quad \text{and} \quad \|\boldsymbol{S}\|_0 \leq s.$$

Non-convexity comes from the *non-convex constraints* (rank and ℓ_0 norm).

Column-wise Data Lost/Corruption

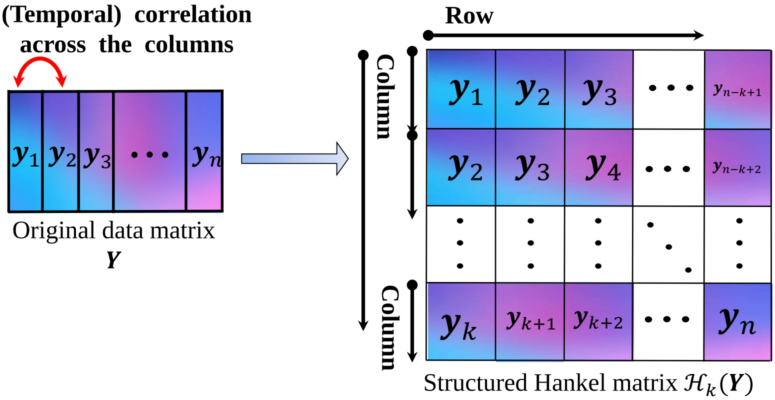
- Recovery needs at least r observations in each column;

- Filling *any linear combination of other columns* would not change the rank;



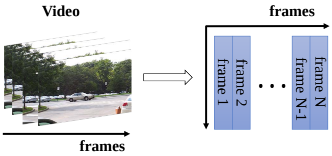
Structured Hankel matrix

- Conduct low-rank matrix completion algorithms on the structured Hankel matrix;

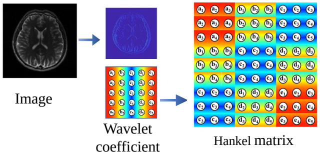


Structured Hankel Matrix

- Video Processing [Ding, et al.07]

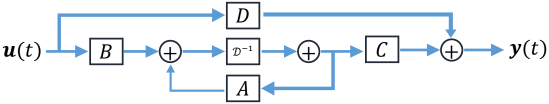


- Magnetic Resonance Imaging [Ongie, et al.16], [Zhang, et al.20]



- Image Super Resolution [Chen et al.14]

- System Identification [Fazel, et al.13]



Robust Low-rank Hankel Matrix Completion

- Objective: given partial observation $\boldsymbol{M}$ in the observation set Ω, decompose it into the sum of two matrices: (1) A low-rank Hankel matrix $\boldsymbol{Y}^*$, and (2) Sparse error $\boldsymbol{S}^*$.

- Solve the non-convex optimization problem:

$$\min_{\boldsymbol{Y},\, \boldsymbol{S}} : \quad \|\mathcal{P}_\Omega(\boldsymbol{M} - \boldsymbol{Y} - \boldsymbol{S})\|_F^2$$

$$\text{s.t.} \qquad \text{rank}\big(\mathcal{H}_\kappa(\boldsymbol{Y})\big) \leq r, \quad \text{and} \quad \|\boldsymbol{S}\|_0 \leq s.$$

$\mathcal{P}_\Omega(Z_{i,j}) = Z_{i,j}$ if the index $(i,j) \in \Omega$ and 0 otherwise.
$\|\boldsymbol{S}\|_0$ is the number of nonzero entries in $\boldsymbol{S}$.

Existing Methods with Guarantees

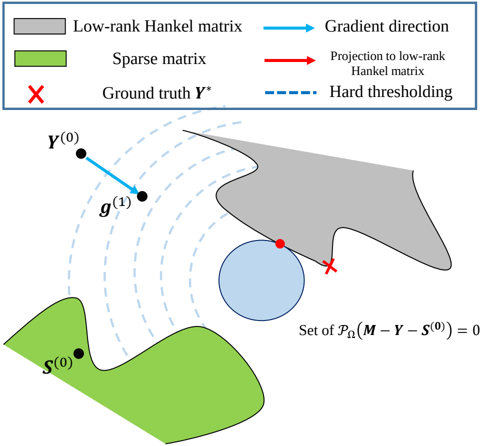
- Single-signal Hankel matrix completion, i.e., $\boldsymbol{S} = 0$, $m = 1$.
  - Hankel matrix nuclear norm relaxation [Fazel et al.10], [Chen et al.14].
  - *FIHT* (non-convex method) through projected gradient descent [Cai et al.17].

Existing Methods with Guarantees

- Single-signal Hankel matrix completion, i.e., $\boldsymbol{S} = 0$, $m = 1$.
  - Hankel matrix nuclear norm relaxation [Fazel et al.10], [Chen et al.14].
  - *FIHT* (non-convex method) through projected gradient descent [Cai et al.17].
- Low-rank matrix recovery, i.e., $\kappa = 1$.
  - *ADMM* (convex approach) through nuclear norm relaxation [Candes et al.11].
  - *SVT* (convex approach) through soft-thresholding on the singular values [Cai et al.10].
  - *R-RMC* (non-convex approaches) through alternative projection [Cherapanamjeri et al.16]

Low-rank methods cannot handle fully lost/corrupted columns.

Our Algorithm: Structured Alternating Projections (SAP)



(S1) Update along the gradient descent:

$$\boldsymbol{g}^{(\ell+1)} = \boldsymbol{Y}^{(\ell)} + p^{-1}\mathcal{P}_\Omega(\boldsymbol{M} - \boldsymbol{Y}^{(\ell)} - \boldsymbol{S}^{(\ell)}).$$

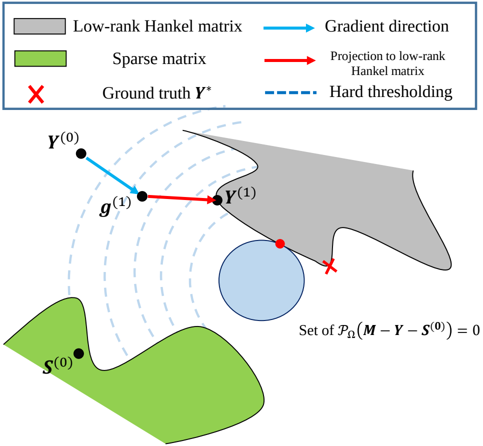
[Zhang et al.JSTSP'18], *[Zhang et al.TSP'19]*

Our Algorithm: Structured Alternating Projections (SAP)

(S2) Project the updated point to the low-rank Hankel matrix space:

$$\mathbf{Y}^{(\ell+1)} = \mathcal{H}_\kappa^\dagger \mathcal{Q}_r(\mathcal{H}_\kappa(\mathbf{g}^{(\ell+1)})).$$

$\mathcal{H}_\kappa^\dagger$: the inverse of $\mathcal{H}_\kappa$.
$\mathcal{Q}_r$: the best rank-r approximation.



[Zhang et al.JSTSP'18], [Zhang et al.TSP'19]

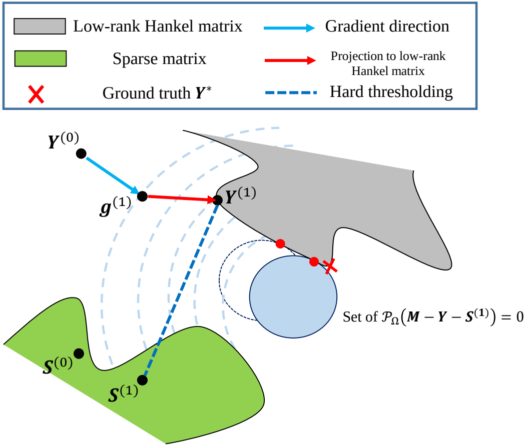
(S3) Update the error matrix via hard thresholding:

$$\xi^{(\ell+1)} = \sigma_{r+1}(\mathcal{H}_\kappa \boldsymbol{g}^{(\ell)}) + \left(\frac{1}{2}\right)^{\ell} \sigma_r(\mathcal{H}_\kappa \boldsymbol{g}^{(\ell)}).$$

$$\boldsymbol{S}^{(\ell+1)} = \mathcal{T}_{\xi^{(\ell+1)}}\left(\mathcal{P}_\Omega(\boldsymbol{M} - \boldsymbol{Y}^{(\ell+1)})\right).$$

$\sigma_k(\cdot)$: the k-th largest singular value.

$$\mathcal{T}_\xi(Z) = \begin{cases} Z, & \text{if } |Z| > \xi. \\ 0, & \text{otherwise.} \end{cases}$$



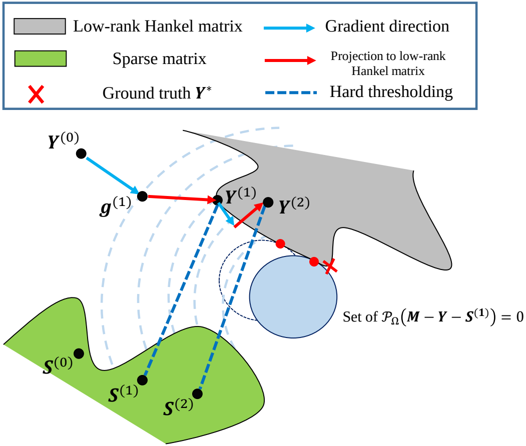
[Zhang et al.JSTSP'18], [Zhang et al.TSP'19]

Our Algorithm: Structured Alternating Projections (SAP)

(S1) Update along the gradient descent:
$$\boldsymbol{g}^{(\ell+1)} = \boldsymbol{Y}^{(\ell)} - p^{-1}\mathcal{P}_\Omega(\boldsymbol{M} - \boldsymbol{Y}^{(\ell)} - \boldsymbol{S}^{(\ell)}).$$

(S2) Project to the low-rank Hankel matrix space:
$$\boldsymbol{Y}^{(\ell+1)} = \mathcal{H}_\kappa^\dagger \mathcal{Q}_r(\mathcal{H}_\kappa(\boldsymbol{g}^{(\ell+1)})).$$

(S3) Update the error matrix via hard thresholding:
$$\xi^{(\ell+1)} = \sigma_{r+1}(\mathcal{H}_\kappa \boldsymbol{g}^{(\ell)}) + \left(\frac{1}{2}\right)^\ell \sigma_r(\mathcal{H}_\kappa \boldsymbol{g}^{(\ell)}).$$
$$\boldsymbol{S}^{(\ell+1)} = \mathcal{T}_{\xi^{(\ell+1)}}(\mathcal{P}_\Omega(\boldsymbol{M} - \boldsymbol{Y}^{(\ell+1)})).$$



[Zhang et al.JSTSP'18], [Zhang et al.TSP'19]

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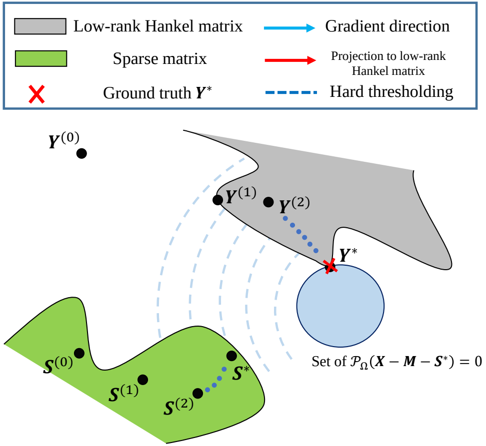


[Zhang et al.JSTSP'18], [Zhang et al.TSP'19]

Comparison with Existing Theoretical Results

A $m \times n$ $(m < n)$ matrix with (Hankel) rank r.

	Low-rank Matrix Recovery		Hankel Matrix Completion		Multi-channel Hankel Matrix Recovery (SAP)
	SVT (convex) [Cai.(10)]	R-RMC (non-convex) [Cherapanamjeri.(17)]	Nuclear norm (convex) [Chen&Chi.(14), Fazel.(10)]	FIHT (non-convex) [Cai.(17)]	
Corruption	Yes		No		Yes
Column-wise lost/corruption	No (Yes on the Hankel matrix)		Not applicable		Yes, up to $1/r$ fraction.
Number of observations	$rn \log n$	$r^3 n \log n$	$r^2 \log n$	$r^3 \log n$	$r^3 \log n$
Computational complexity	rn^2/ε (rn^3/ε on Hankel)	$r^3 n \log(1/\varepsilon)$ ($r^3 n^2 \log(1/\varepsilon)$ on Hankel)	$r^2 n/\varepsilon$	$r^3 n \log(1/\varepsilon)$	$r^3 n \log(1/\varepsilon)$

[Zhang et al.JSTSP'18], [Zhang et al.TSP'19]

Theoretical Results [1]

Theorem 1

Suppose the following conditions hold:
(1) $\boldsymbol{S}^*$ *contains at most* $\mathcal{O}(1/r)$ *of fully corrupted columns;*
(2) required observations (random sampling): $|\Omega| \geq \Theta\big(r^3 \log t \log(1/\varepsilon)\big)$.
Then, with high probability, $\|\boldsymbol{Y}^{(L)} - \boldsymbol{Y}^*\|_F \leq \varepsilon$ *with* $L = \log(1/\varepsilon)$.

- SAP can tolerate up to a constant fraction of fully corrupted or lost columns; the conventional matrix completion algorithms fails with only one fully corrupted or lost column;
- The required number of samples is $\Theta(r^3 \log n)$;
- The algorithm enjoys a linear convergence rate, and the computational complexity is $\Theta(r^2 n \log n \log(1/\varepsilon))$.

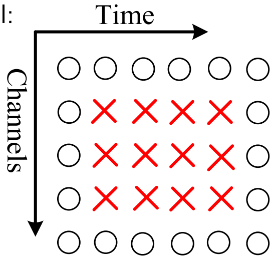
[1] *[Zhang et al.JSTSP'18]*, *[Zhang et al.TSP'19]*

Evaluation on Synthetic Data

- Data model: m signals, each is a weighted sum of r sinusoids.

$$Y_{k,t} = \sum_{i=1}^{r} d_{k,i} e^{-\tau_i t} e^{j(2\pi f_i + \phi_i)t}.$$

- $\boldsymbol{Y}$ is in 20×600, and r is 15.
- Data loss model:



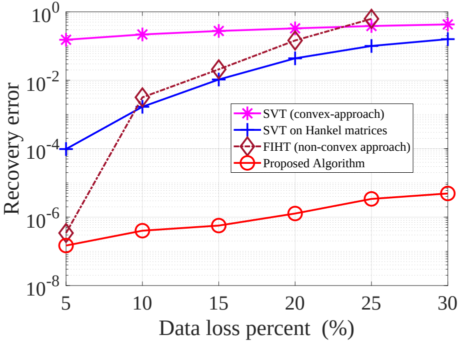


Figure 14: The recovery error of recovered data with simultaneous and consecutive data loss.

Simulation Results: Synthetic Data

The computational time for recovering a matrix in $20 \times t$; for achieving a recovery error ε, the computational time of

- SAP (our proposed algorithm) is in the order of $t \log t \log(1/\varepsilon)$;
- R-RMC on Hankel matrix is in the order of $t^2 \log t \log(1/\varepsilon)$;
- ADMM on Hankel matrix is at least in the order of t^3/ε;

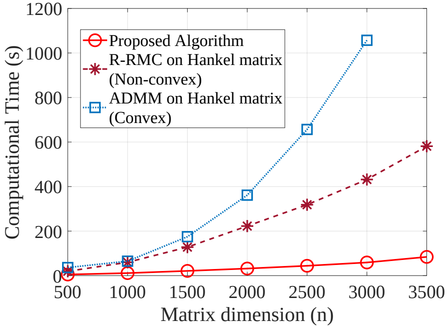


Figure 15: Comparison of computational time with conventional low-rank matrix completion approaches

Evaluation on Synchrophasor Data

Randomly located missing and bad data.

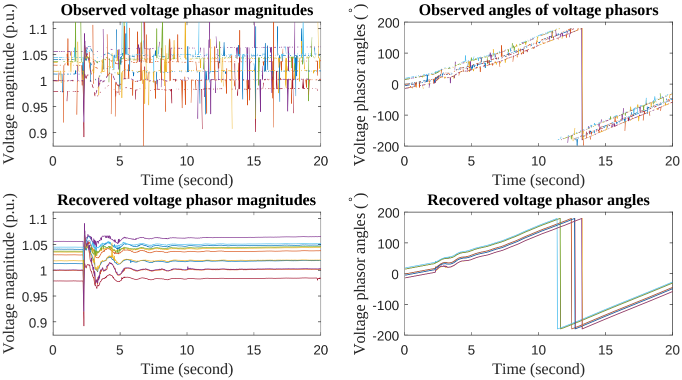


Figure 16: One case of 8% random bad data and 40% random missing data

Consecutive bad data are corrected. Event disturbance is maintained.

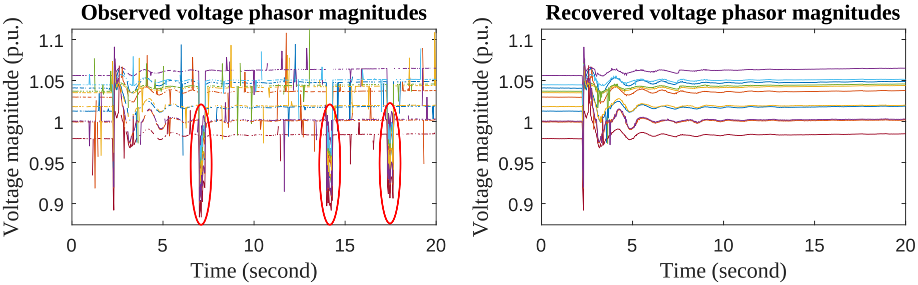


Figure 17: Consecutive bad data, 3% random bad data and 20% missing data

Proof Sketch of SAP

The update rule of $\boldsymbol{Y}^{(\ell+1)}$ is

$$\mathcal{H}\boldsymbol{Y}^{(\ell+1)} = \mathcal{Q}_r\mathcal{H}\left(\boldsymbol{Y}^* + (\mathcal{I} - p^{-1}\mathcal{P}_\Omega)(\boldsymbol{Y}^{(\ell)} + \boldsymbol{S}^{(\ell)} - \boldsymbol{Y}^* - \boldsymbol{S}^*) + (\boldsymbol{S}^{(\ell)} - \boldsymbol{S}^*)\right).$$

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We have $\|\boldsymbol{Y}^{(\ell+1)} - \boldsymbol{Y}^*\|_\infty \leq \frac{1}{2}\|\boldsymbol{Y}^{(\ell)} - \boldsymbol{Y}^*\|_\infty$ if $|\Omega|$ is sufficiently large and α is sufficiently small.

α: fraction of non-zero entries in $\boldsymbol{S}^{(\ell)} - \boldsymbol{S}^*$; upper bounded by the fraction of corrupted columns or entries as we prove that the support of $\boldsymbol{S}^{(\ell)} - \boldsymbol{S}^*$ is always a subset of $\boldsymbol{S}^*$.

Proof Sketch of SAP

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Let $\boldsymbol{U}$ be space of $\mathcal{H}\boldsymbol{Y}^{(\ell+1)}$, we have

$$\begin{aligned}\|\mathcal{H}\boldsymbol{Y}^{(\ell+1)} - \mathcal{H}\boldsymbol{Y}^*\|_\infty =& \sum_{i=1}^{r}\|(\mathcal{H}\boldsymbol{Y}^{(\ell+1)} - \mathcal{H}\boldsymbol{Y}^*)\boldsymbol{u}_i\|_\infty + \|(\mathcal{I} - \mathcal{P}_{\boldsymbol{U}})\mathcal{H}\boldsymbol{Y}^*\|_\infty \\ \leq& r \cdot \|\boldsymbol{E}_1 + \boldsymbol{E}_2\|_2 \cdot \|\boldsymbol{Y}^{(\ell)} - \boldsymbol{Y}^*\|_\infty + \|(\mathcal{I} - \mathcal{P}_{\boldsymbol{U}})\mathcal{H}\boldsymbol{Y}^*\|_\infty.\end{aligned}$$

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By projecting into a rank-r Hankel matrix space, the order of first item is reduced from t to r, however, at a cost of a projection error (second item).

Proof Sketch of SAP

The update rule of $\boldsymbol{Y}^{(\ell+1)}$ is

$$\mathcal{H}\boldsymbol{Y}^{(\ell+1)} = \mathcal{Q}_r\mathcal{H}\left(\boldsymbol{Y}^* + \underbrace{(\mathcal{I} - p^{-1}\mathcal{P}_\Omega)(\boldsymbol{Y}^{(\ell)} + \boldsymbol{S}^{(\ell)} - \boldsymbol{Y}^* - \boldsymbol{S}^*)}_{\boldsymbol{E}_1 \text{ in the order of } \log n/\sqrt{|\Omega|}} + \underbrace{(\boldsymbol{S}^{(\ell)} - \boldsymbol{S}^*)}_{\boldsymbol{E}_2 \text{ in the order of } \alpha}\right).$$

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To guarantee the convergence, we need

$$\|\boldsymbol{E}_1 + \boldsymbol{E}_2\|_2 \leq \frac{1}{2r} \Longrightarrow \Omega > \Theta(r^2\log^2(n)) \text{ and } \alpha > \Theta(1/r),$$

and $\|(\mathcal{I} - \mathcal{P}_{\boldsymbol{U}})\mathcal{H}\boldsymbol{Y}^*\|_\infty$ is in the order of $\|\boldsymbol{Y}^{(\ell)} - \boldsymbol{Y}^*\|_\infty$ (major technique challenges).